Computable Categoricity for CSC Spaces

Andrew DeLapo

University of Connecticut

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Background

Computable categoricity is a notion from computable structure theory.

Definition

A structure $\mathcal{A}$ is **computably categorical** if for every computable structure $\mathcal{B} \cong \mathcal{A}$, there is a computable isomorphism $\mathcal{A} \to \mathcal{B}$.

This has been studied for algebraic, combinatorial, and even some topological structures.

CSC Spaces

Definition (Dorais 2011)

A **countable second-countable space** (**CSC space**) is a triple $(X, \mathcal{U}, k)$ where X is a countable set, $\mathcal{U} = (U_i)_{i\in\omega}$ is a countable basis for open sets in X, and k is a function $X \times \omega \times \omega \to \omega$ such that

- for all $x \in X$, there is $i \in \omega$ such that $x \in U_i$,
- for all $x \in X$ and $i, j \in \omega$, if $x \in U_i \cap U_j$, then $x \in U_{k(x,i,j)} \subseteq U_i \cap U_j$.

CSC spaces provide an excellent context for studying topological facts in computability theory and reverse mathematics (Dorais 2011, Shafer 2020, Benham, D., Dzhafarov, Solomon, and Villano 2024, Genovesi 2024, DeLapo and Gonzalez 2025).

CSC Spaces

Definition

A **computable CSC space** is a CSC space $(\omega, \mathcal{U}, k)$ where $\mathcal{U} = (U_i)_{i \in \omega}$ is uniformly computable and k is computable.

Example

The discrete topology on ω has a presentation as a computable CSC space: $\mathbb{N}_{DIS} = (\omega, \mathcal{U}, k)$ where $U_i = \{i\}$ for all i and $k(x, i, j) = i$.

Definition

A CSC space $(X, \mathcal{U}, k)$ is a **computable copy** of $(Y, \mathcal{V}, \ell)$ if X is a computable CSC space and $X \cong Y$.

Notice that computable CSC spaces appear to have no presentation as structures in the traditional sense.

Four Motivating Examples

We will consider the following examples.

- $\mathbb{N}_{IND}$ is the CSC space $(\omega, \mathcal{V}, k)$ with $V_i = \omega$ for all i. Then $\mathbb{N}_{IND}$ has the **indiscrete topology**.
- $\mathbb{N}_{DIS}$ is the CSC space $(\omega, \mathcal{V}, k)$ with $V_i = \{i\}$ for all i. Then $\mathbb{N}_{IND}$ has the **discrete topology**.
- $\mathbb{N}_{COF}$ is the CSC space $(\omega, \mathcal{V}, k)$ with $V_i = \omega \setminus D_i$ for all i, where D_i is the finite set coded by i. Then $\mathbb{N}_{COF}$ has the **cofinite topology**.
- $\mathbb{N}_{IST}$ is the CSC space $(\omega, \mathcal{V}, k)$ with $V_i = [0, i]$ for all i. Then $\mathbb{N}_{IND}$ has the **initial segment topology**.

Motivation

Theorem (Ginsburg and Sands 1979)

Every infinite topological space X has an infinite subspace homeomorphic to one of the following five toplogies on ω:

1. *Indiscrete topology* $\mathbb{N}_{IND}$
2. *Initial segment topology* $\mathbb{N}_{IST}$
3. *Final segment topology*
4. *Cofinite topology* $\mathbb{N}_{COF}$
5. *Discrete topology* $\mathbb{N}_{DIS}$

Computable Categoricity

Definition

Let X be a CSC space.

- The **weak computable dimension** of X is the number of computable copies of X up to computable homeomorphism.
- X is **weakly computably categorical** if X has weak computable dimension 1.

Proposition

$\mathbb{N}_{IND}$, $\mathbb{N}_{DIS}$, and $\mathbb{N}_{COF}$ are weakly computably categorical.

Proof.

Let X be a computable discrete CSC space. Any computable bijection $X \to \mathbb{N}_{DIS}$ is a computable homeomorphism. The analogous facts hold for $\mathbb{N}_{IND}$ and $\mathbb{N}_{COF}$. $\square$

Effective Homeomorphisms

Definition (Dorais 2011)

Let $(X, \mathcal{U}, k)$ and $(Y, \mathcal{V}, \ell)$ be CSC spaces.

- A function $f : X \to Y$ is **effectively continuous** if f is computable and there is a computable function Φ such that for all x and i, if $f(x) \in V_i$, then $x \in U_{\Phi(x,i)} \subseteq f^{-1}(V_i)$.
- A function $f : X \to Y$ is an **effective homeomorphism** if f is a bijection and both f and f^{-1} are effectively continuous.

Definition

Let X be a CSC space.

- The **computable dimension** of X is the number of computable copies of X up to effective homeomorphism.
- X is **computably categorical** if X has computable dimension 1.

Why *Effective* Homeomorphisms?

1. Quantifiers:
   - For computable structures $\mathcal{A}$ and $\mathcal{B}$, the statement "$\mathcal{A}$ and $\mathcal{B}$ are computably isomorphic" is Σ^0_3.
   - For computable CSC spaces X and Y, the statement "X and Y are computably homeomorphic" is Σ^0_4.
   - For computable CSC spaces X and Y, the statement "X and Y are effectively homeomorphic" is Σ^0_3.
2. Effective homeomorphisms preserve effective properties.

Discrete Topology

Definition

- For a CSC space $(X, \mathcal{U}, k)$, a **discreteness function** for X is a function $d : X \to \omega$ such that $U_{d(x)} = \{x\}$ for all $x \in X$.
- A CSC space is **effectively discrete** if it has a computable discreteness function.

Lemma

Let X and Y be effectively homeomorphic CSC spaces, and let d_X be a discreteness function for X. Then Y has a discreteness function d_Y computable from d_X. In particular, if X is effectively discrete, then so is Y.

Discrete Topology

Fact

There are computable discrete CSC spaces which are not effectively discrete (see e.g. Dorais 2011 and Benham, D., Dzhafarov, Solomon, and Villano 2024).

Proposition

$\mathbb{N}_{DIS}$ is not computably categorical.

Theorem

For each $e \in \omega$, there is a computable discrete CSC space X_e such that X_e has a unique discreteness function d_e, and $d_e \equiv_T W_e$.

Corollary

$\mathbb{N}_{DIS}$ has computable dimension ω.

Cofinite Topology

Lemma

Let $X = (\omega, \mathcal{U}, k) \cong \mathbb{N}_{COF}$*.*

- *The function* $i \mapsto |\omega \setminus U_i|$ *computes an effective homeomorphism* $X \to \mathbb{N}_{COF}$*.*
- *Any effective homeomorphism* $X \to \mathbb{N}_{COF}$ *computes the function* $i \mapsto |\omega \setminus U_i|$*.*

Theorem

For each $e \in \omega$*, there is a computable CSC space* $X_e \cong \mathbb{N}_{COF}$ *such that the function* $i \mapsto |\omega \setminus U_i|$ *has the same Turing degree as* W_e*.*

Corollary

$\mathbb{N}_{COF}$ has computable dimension ω.

Initial Segment Topology

Fact

If X is homeomorphic to $\mathbb{N}_{IST}$, then the homeomorphism is unique.

Theorem

For each e, there is a computable CSC space X_e such that X_e has the initial segment topology, and the Turing degree of the unique homeomorphism $X_e \to \mathbb{N}_{IST}$ is the same as that of W_e.

Corollary

$\mathbb{N}_{IST}$ has weak computable dimension ω.

Initial Segment Topology

Theorem

For each e, there is a computable CSC space X_e such that X_e has the initial segment topology, and the Turing degree of the unique homeomorphism $X_e \to \mathbb{N}_{IST}$ is the same as that of W_e.

Proof sketch.

Fix e, and assume WLOG that W_e is infinite and coinfinite. Write x_s for the element of W_e enumerated at stage s. For all s, let ℓ_s be the least ℓ such that $[\ell, x_s] \subseteq W_{e,s}$. For all $i \in \omega$, let

$$U_i = \{s : \ell_s < \ell_i \text{ or } (\ell_s = \ell_i \text{ and } x_s \leq x_i)\}$$

and let $X_e = (\omega, \mathcal{U}, k)$ be the resulting CSC space. $\square$

Categoricity Results

In summary:

CSC Space	**Weak Computable Dim.**	**Computable Dim.**
$\mathbb{N}_{IND}$	1	1
$\mathbb{N}_{DIS}$	1	ω
$\mathbb{N}_{COF}$	1	ω
$\mathbb{N}_{IST}$	ω	ω

Further Questions

1. For each $n > 1$, is there a computable CSC space with (weak) computable dimension n?
2. Investigate categoricity for other CSC spaces:
   - $\mathbb{Q}$
   - CSC spaces arising from linear orders
3. Computable categoricity relative to a degree, degree spectra, ...

Bonus Result

Question

Is there a CSC space with no computable presentation?

Definition

Let α be a countable ordinal. Write α_{IST} for the CSC space $(\alpha, (\beta)_{\beta<\alpha}, k)$ where $k(x, i, j) = \min(i, j)$.

Proposition

If α is a noncomputable ordinal, then α_{IST} has no computable presentation.

Proof idea.

Suppose $(\omega, \mathcal{U}, k) \cong \alpha_{IST}$. For $a, b \in \omega$, say that $a <_\alpha b$ if and only if $\exists i(a \in U_i \wedge b \notin U_i)$. Then $(\omega, <_\alpha) \cong \alpha$ as linear orders.

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