Computable Categoricity of Countable Second-Countable Spaces

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Graduate Research Forum
January 25, 2025

Motivation

Let X and Y be homeomorphic topological spaces.

- How complicated is a homeomorphism $f : X \to Y$? Is there a **computable** homeomorphism?
- If not, how many computable copies of X exist up to computable homeomorphism?

Computability Background

Definition

A partial function $f : \mathbb{N} \to \mathbb{N}$ is **computable** if there is an algorithm (e.g. a computer program) P which can do what f does.

- If $f(x) = y$, then P should take input x and return output y in some finite amount of time.
- If f is undefined on input x, then P should take input x and loop forever, never returning any output.

Definition

A set $A \subseteq \mathbb{N}$ is **computable** if its characteristic function χ_A is computable. In other words, there is an algorithm P such that:

- if $x \in A$, then P takes input x and returns output 1,
- if $x \notin A$, then P takes input x and returns output 0.

Computability Background

Fact

There are countably many programs, so there are countably many computable functions.

Theorem (MRDP 1970)

In general, the function $f : \mathbb{N}^k \to \mathbb{N}$ *given by*

$$f(a_1, \ldots, a_k) = \begin{cases} 1 & \textit{if } a_1 x_1 + \cdots + a_k x_k = 0 \textit{ has an integer solution} \\ 0 & \textit{otherwise} \end{cases}$$

is not computable, answering Hilbert's 10th question.

CSC Spaces

Definition (Dorais 2011)

A **countable second-countable space** (**CSC space**) is a triple $(X, \mathcal{U}, k)$ where X is a countable set, $\mathcal{U} = (U_i)_{i \in \mathbb{N}}$ is a countable basis for open sets in X, and k is a function $X \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that

- for all $x \in X$, there is $i \in \mathbb{N}$ such that $x \in U_i$,
- for all $x \in X$ and $i, j \in \mathbb{N}$, if $x \in U_i \cap U_j$, then $x \in U_{k(x,i,j)} \subseteq U_i \cap U_j$.

CSC spaces provide an excellent context for studying topological facts in computability theory and reverse mathematics (Dorais 2011, Shafer 2020, Benham et al. 2024).

CSC Spaces

Definition

A **computable CSC space** is a CSC space $(\mathbb{N}, \mathcal{U}, k)$ where $\mathcal{U} = (U_i)_{i \in \mathbb{N}}$ is uniformly computable and k is computable. That is, there is a computable function $f : \mathbb{N}^2 \to \mathbb{N}$ such that

$$f(x, i) = \begin{cases} 1 & \text{if } x \in U_i \\ 0 & \text{if } x \notin U_i \end{cases}.$$

Example

The discrete topology on $\mathbb{N}$ has a presentation as a computable CSC space: $\mathbb{N}_{DIS} = (\mathbb{N}, \mathcal{U}, k)$ where $U_i = \{i\}$ for all i and $k(x, i, j) = i$.

Effective Homeomorphisms

Definition (Dorais 2011)

Let $(X,\mathcal{U},k)$ and $(Y,\mathcal{V},\ell)$ be CSC spaces.

- A function $f : X \to Y$ is **effectively continuous** if f is computable and there is a computable function Φ such that for all x and i, if $f(x) \in V_i$, then $x \in U_{\Phi(x,i)} \subseteq f^{-1}(V_i)$.
- A function $f : X \to Y$ is an **effective homeomorphism** if f is a bijection and both f and f^{-1} are effectively continuous.

Indiscrete Topology

Definition

A (CSC) topological space X has the **indiscrete topology** if the only open sets in X are $\emptyset$ and X.

Example

Write $\mathbb{N}_{IND}$ for the CSC space $(\mathbb{N}, \mathcal{U}, k)$ where $U_i = \mathbb{N}$ for all i, and $k(x, i, j) = i$. Then $\mathbb{N}_{IND}$ is a computable CSC space with the indiscrete topology.

Question

Given a computable CSC space X with the indiscrete topology, how hard is it to *find* an effective homeomprhism $f : X \to \mathbb{N}_{IND}$?

Indiscrete Topology

Proposition

If X is a computable CSC space with the indiscrete topology, then the identity map is an effective homeomorphism from X to $\mathbb{N}_{IND}$.

Definition

A CSC space X is **computably categorical** if for every computable CSC space Y homeomorphic to X, there is an *effective* homeomorphism $Y \to X$.

Hence we say the indiscrete topology is **computably categorical**.

Discrete Topology

Definition

A (CSC) topological space X has the **discrete topology** if every subset of X is open in X.

Example

Write $\mathbb{N}_{DIS}$ for the CSC space $(\mathbb{N}, \mathcal{U}, k)$ where $U_i = \{i\}$ for all i, and $k(x, i, j) = i$. Then $\mathbb{N}_{DIS}$ is a computable CSC space with the discrete topology.

If X is a computable CSC space with the discrete topology, every computable bijection $X \to \mathbb{N}_{DIS}$ is a homeomorphism. When is the homeomorphism *effective*?

Discrete Topology

Definition (Dorais 2011)

A CSC space $(X, \mathcal{U}, k)$ is **effectively discrete** if there is a computable function $d : X \to \mathbb{N}$ such that $U_{d(x)} = \{x\}$ for all $x \in X$.

Facts

- There exist computable CSC spaces which are discrete but not effectively discrete (see Dorais 2011 or Benham et al. 2024).
- If X and Y are effectively homeomorphic CSC spaces and X is effectively discrete, then so is Y.

Proposition

$\mathbb{N}_{DIS}$ is *not* computably categorical.

Initial Segment Topology

Example

Write $\mathbb{N}_{IST}$ for the CSC space $(\mathbb{N}, \mathcal{U}, k)$ with $U_n = [0, n]$ for all $n \in \mathbb{N}$, and $k(x, i, j) = \min(i, j)$. See that $\mathbb{N}_{IST}$ is a computable CSC space.

Definition

A (CSC) topological space X has the **initial segment topology** if X is homeomorphic to $\mathbb{N}_{IST}$.

Unlike with $\mathbb{N}_{IND}$ and $\mathbb{N}_{DIS}$, if there is a homeomorphism $f : X \to \mathbb{N}_{IST}$, then f is *unique*.

Initial Segment Topology

For each $e \in \mathbb{N}$, let Φ_e denote the eth computable function, and write $W_e = \{x \in \mathbb{N} : \Phi_e(x) \downarrow\}$. In general, W_e is not computable.

Theorem

For each e such that W_e is noncomputable, there is a computable CSC space X_e such that X_e has the initial segment topology, and the Turing degree of the unique homeomorphism $X_e \to \mathbb{N}_{IST}$ is the same as that of W_e.

Corollary

$\mathbb{N}_{IST}$ is not computably categorical.

Further Questions

Investigate computable categoricity for other CSC spaces:

- Cofinite topology on $\mathbb{N}$
- $\mathbb{Q}$ with the Euclidean topology

Definition

The **computable dimension** of a CSC space X is the number of computable copies of X up to effective homeomorphism.

We showed $\mathbb{N}_{IND}$ has computable dimension 1, and we can show $\mathbb{N}_{IST}$ and $\mathbb{N}_{DIS}$ have infinite computable dimension. Does there exist a CSC space with finite computable dimension greater than 1?

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