Index Sets of CSC Spaces

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Motivation

Index sets of classes and subclasses of structures have been studied in several contexts:

- Groups, see e.g. Calvert 2005, Knight and Saraph 2017
- Fields, see e.g. Calvert 2004
- Polish spaces, see Thewmorakot 2023

I am interested in classes of topological spaces with fundamental topological properties:

- Separation axioms (T_0, T_1, Hausdorff)
- Homeomorphic to common topological spaces (cofinite topology, discrete topology)
- Compactness, connectedness, metrizability

CSC Spaces

To code topological spaces with ω, we can restrict to topological spaces with countably many points and a countable basis of open sets.

Definition (Dorais 2011)

A **countable, secound countable (CSC) space** is a triple $(X, \mathcal{U}, k)$ where X is a countable set, $\mathcal{U}$ is a countable sequence $\mathcal{U} = (U_i)_{i \in \omega}$ of subsets of X, and k is a function $k : X \times \omega \times \omega \to \omega$, such that

- for all $x \in X$, there is $i \in \omega$ such that $x \in U_i$
- for all $x \in X$ and $i, j \in \omega$, if $x \in U_i \cap U_j$, then $x \in U_{k(x,i,j)} \subseteq U_i \cap U_j$.

CSC spaces provide an excellent context for studying topological facts in reverse mathematics (Dorais 2011, Shafer 2020, Benham et al. 2024). For the remainder of this talk, CSC spaces will have domain ω.

CSC Spaces

Definition

A CSC space $(\omega, \mathcal{U}, k)$ is **computable** if $\mathcal{U}$ is uniformly computable and k is computable. That is, there are indices m and n such that Φ_m and Φ_n are total, $k = \Phi_n$, and

$$x \in U_i \iff \Phi_m(i, x) = 1$$

for all $i, x \in \omega$.

Then $\langle m, n \rangle$ is an **index** for a CSC space. Write

$$CSC = \{e : e \text{ is an index for a CSC space}\}.$$

Theorem (D.)

The set CSC is Π^0_2-complete.

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Notation

Let $e = \langle m, n \rangle$ be an index for a CSC space.

- Write "$x \in U_i$" for $\Phi_m(i, x) = 1$.
- Write "$x \in U_i \cap U_j$" for $\Phi_m(i, x) = 1 \wedge \Phi_m(j, x) = 1$.

Example

Let T_2-$CSC = \{e : e \text{ is an index for a Hausdorff CSC space}\}$. The T_2 axiom can be written

$$(\forall x \neq y) \exists i \exists j \forall z (x \in U_i \wedge y \in U_j \wedge z \notin U_i \cap U_j)$$

so we expect T_2-CSC to be Π^0_3-complete.

Many-One Reductions

Definition

A set B is **many-one reducible** to a set A, written $B \leq_m A$, if there is a computable function f such that

$$x \in B \iff f(x) \in A$$

for all $x \in \omega$.

Definition

Let Γ be a complexity class.

- A set A is **Γ-hard** if $B \leq_m A$ for all $B \in \Gamma$.
- A set A is **Γ-complete** if $A \in \Gamma$ and A is Γ-hard.

Many-One Reductions *within* a Set

Definition (Calvert 2005, Knight)

Let Γ be a complexity class, let I be a set, and let A be a set.

- The set A is **Γ-within** I if $A = B \cap I$ for some $B \in \Gamma$.
- The set A is **Γ-hard within** I if for every $B \in \Gamma$, there is a computable function f such that

$$x \in B \iff f(x) \in A$$

  and $f(x) \in I$, for all $x \in \omega$.
- The set A is **Γ-complete within** I if A is Γ-within I and Γ-hard within I.

Example

The set $T_2\text{-}CSC = \{e : e \text{ is an index for a Hausdorff CSC space}\}$ is Π^0_3 within CSC.

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Theorem (D.)

The set T_2-CSC $= \{e : e$ is an index of a Hausdorff CSC space$\}$ is Π^0_3-complete within CSC.

Proof.

It remains to show T_2-CSC is Π^0_3-hard within CSC. Recall that CoInf $= \{e : W_e$ is coinfinite$\}$ is Π^0_3-complete. Fix e. For all x and y, let

$$V_{\langle x,y\rangle} = \begin{cases} \{x\} \cup \{s : \Phi_{e,s}(y)\downarrow\} & \text{if } y \geq x \\ \omega & \text{otherwise} \end{cases}$$

and let X_e be the resulting CSC space. Suppose $e \in$ CoInf, and let $x_0 < x_1$. There is $y \geq x_1$ such that for all s, $\Phi_{e,s}(y)\uparrow$, so $V_{\langle x_0,y\rangle} = \{x_0\}$ and $V_{\langle x_1,y\rangle} = \{x_1\}$. Hence X_e is Hausdorff. If $e \notin$ CoInf, then fix x such that for all $y \geq x$, $\Phi_e(y)\downarrow$. In particular, every open set containing x or $x+1$ is cofinite, so X_e cannot be Hausdorff. □

Corollary

The set Disc-CSC $= \{e : e$ *is an index for a discrete CSC space*$\}$ *is* Π^0_3*-complete within CSC.*

Proof.

The set

$$B = \{\langle m, n\rangle : \forall x \exists i \forall y (y \in U_i \leftrightarrow y = x)\}$$

is Π^0_3, and $Disc\text{-}CSC = B \cap CSC$, so $Disc\text{-}CSC$ is Π^0_3-within CSC.

The space constructed in the previous proof was also discrete if and only if it was Hausdorff, so the previous proof shows $Disc\text{-}CSC$ is Π^0_3-complete within CSC. $\square$

Separate Hausdorffness from discreteness?

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Theorem (D.)

The set Disc-CSC is Π^0_3*-complete within* T_2*-CSC.*

Proof idea.

Let $e \in \omega$. The space X_e we construct must always be Hausdorff, but not discrete unless $e \in \mathrm{CoInf}$. For all $x < y$, define

$$\begin{aligned} V_{2\langle x,y\rangle} &= \{t \in \omega : t \equiv x \bmod y\} \\ V_{2\langle y,x\rangle} &= \{t \in \omega : t \equiv 0 \bmod y\} \\ V_{2\langle x,y\rangle+1} &= \{x\} \cup \{s : \Phi_{e,s}(y)\downarrow\} \end{aligned}$$

and for all other V_i not specified above, let $V_i = \emptyset$. An elementary number theory argument shows that the collection $(V_{2i})_{i\in\omega}$ is closed under finite intersection. The resulting CSC space X_e has the desired properties. $\square$

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Theorem (D.)

- *The set Indiscrete-CSC is* Π^0_1*-complete within CSC.*
- *The set* T_0*-CSC is* Π^0_2*-complete within CSC.*
- *The set* T_1*-CSC is* Π^0_2*-complete within* T_0*-CSC.*
- *The set* T_2*-CSC is* Π^0_3*-complete within* T_1*-CSC.*
- *The set* Cof*-CSC is* Π^0_3*-complete within* T_1*-CSC.*

Theorem (D.)

Let ω_{IST} *be the CSC space* $(\omega, ([0,n])_{n\in\omega}, k)$. *The set*

$$\left\{ e : \begin{array}{l} e \textit{ is an index for a CSC space which has} \\ \textit{a subspace homeomorphic to } \omega_{IST} \end{array} \right\}$$

is Σ^1_1*-complete within CSC.*

Index Sets of CSC Spaces: Future Work

Conjectures

- $IST\text{-}CSC$ is Π^0_3-complete within CSC.
- The set $\{\langle i, j\rangle : i$ and j are indices of homeomorphic CSC spaces$\}$ is Σ^1_1-complete.

Questions

What are the complexities of the following sets?

- $\{e : e$ is an index of a compact CSC space$\}$
- $\{e : e$ is an index of a connected CSC space$\}$
- $\{e : e$ is an index of a metrizable CSC space$\}$
- $\{e : e$ is an index of a $[T_3, T_4, ...]$ CSC space$\}$
- $\{\langle i, j\rangle : i$ and j are indices of computably homeomorphic CSC spaces$\}$

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Appendix

Suppose we want to show some index set A is Γ-hard within CSC. We will argue as follows. Choose a Γ-complete set B. The goal is to find computable functions $m(x)$ and $n(x)$ such that $e \mapsto \langle m(e), n(e) \rangle$ is the desired many-one reduction within CSC.

- Define, uniformly in $e \in \omega$, a uniformly computable sequence $\mathcal{V}^e = (\mathcal{V}_i^e)_{i \in \omega}$ of sets, so that $\mathcal{V} = (\mathcal{V}^e)_{e \in \omega}$ is given by a computable function $h(e, i, x)$.
- Close each $\mathcal{V}^e$ under finite intersection by letting $U^e_{\langle i_0, \ldots, i_\ell \rangle} = \bigcap_{n=0}^{\ell} V^e_{i_n}$. The $\mathcal{U}^e$ are given by

$$h'(e, \langle i_0, \ldots, i_\ell \rangle, x) = \prod_{s=0}^{\ell} h(e, i_s, x)$$

$$k'(e, x, \langle i_0, \ldots, i_\ell \rangle, \langle j_0, \ldots, j_{\ell'} \rangle) = \langle i_0, \ldots, i_\ell, j_0, \ldots, j_{\ell'} \rangle.$$

- By the s-m-n theorem, there are total functions m and n such that $\Phi_{m(e)}(i, x) = h'(e, i, x)$ and $\Phi_{n(e)}(x, i, j) = k'(e, x, i, j)$.

Appendix

- For each e, we see that $X_e = (\omega, \mathcal{U}^e, \Phi_{q(e)})$ is a computable CSC space with index $\langle m(e), n(e) \rangle$. Let f be the function $e \mapsto \langle m(e), n(e) \rangle$. Then $f(e) \in CSC$ for all e.
- If the $\mathcal{V}^e = (V_i^e)_{i \in \omega}$ are chosen wisely, then X_e will have the desired topological property defined in A if and only if $e \in B$. That means we will have

$$e \in B \iff f(e) \in A$$

in order to prove that A is Γ-hard within CSC.

Summary

For a fixed e, define a sequence $\mathcal{V} = (V_i^e)_{i \in \omega}$ that becomes (in the way defined above) a CSC space $X_e = (\omega, \mathcal{U}, k)$. Then, argue that X_e has the desired topological property if and only if $e \in B$.