Topics in Logic at UConn

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1 Introduction

Almost every semester, the Mathematics Department at the University of Connecticut offers the graduate-level course MATH 5026: Topics in Mathematical Logic. The main topic of focus changes each semester. The goal of this document is to summarize the main ideas, results, and proofs from each course.

Any mistakes in these notes are likely transcription errors and are entirely the fault of the author.

2 Computability, Randomness, and Geometry (Fall 2021)

This course was taught by Professor Reed Solomon.

2.1 Computability

2.1.1 Register Machines

Definition 2.1. A **register machine** M consists of a finite set of registers $R_0, R_1, \ldots, R_n$, each holding a natural number, and a finite list of instructions $L_0, L_1, \ldots, L_\ell$, each in one of the following forms:

- $\mathtt{R_i} := \mathtt{R_i} + \mathtt{1}$.
- `HALT.`
- If $\mathtt{R_i} \neq 0$ then $\mathtt{R_i} := \mathtt{R_i} - 1$ and $\mathtt{L_a}$, else $\mathtt{L_b}$.

We can run such a register machine M on input $a_0, a_1, \ldots, a_k \in \mathbb{N}$ (where $k \leq n$) by setting $R_i = a_i$ for each $i \leq k$, $R_i = 0$ for the remaining $k < i \leq n$, and following the instructions beginning with L_0. If we reach a `HALT` instruction with $R_0 = b$, then write $M(a_0, \ldots, a_k) \downarrow = b$ and say the computation **converges**. If no `HALT` instruction is ever reached, then write $M(a_0, \ldots, a_k) \uparrow$ and say the computation **diverges**.

Definition 2.2. A **partial function** $f : X \to Y$ is a function whose domain is a subset of X.

- A partial function $f : \mathbb{N}^k \to \mathbb{N}$ is **partial RM-computable** if there is a register machine M with registers $R_0, \ldots, R_n$ with $n \geq k - 1$ such that for all $a_0, \ldots, a_{k-1} \in \mathbb{N}$,

$$f(a_0, \ldots, a_{k-1}) \simeq M(a_0, \ldots, a_{k-1})$$

where $\simeq$ means the values are equal if both computations converge, or both diverge.

- If $f : \mathbb{N}^k \to \mathbb{N}$ is partial RM-computable and $\mathrm{dom}(f) = \mathbb{N}^k$, then f is **(total) RM-computable**.
- A set (or relation) $R \subseteq \mathbb{N}^k$ is **computable** if its characteristic function is computable.

From now on, we just write "computable" for "RM-computable."

Fact. The class of partial computable functions is closed under composition, primitive recursion, and μ-recursion.

2.1.2 Coding Sequences

Recall that $\mathbb{N}^k$ and $\mathbb{N}$ are in bijection. Often we will think of a finite tuple $(a_0, a_1, \ldots, a_k)$ as being "coded" by a single finite number via such a bijection. A convenient (and computable) bijection $\langle \cdot, \cdot \rangle : \mathbb{N}^2 \to \mathbb{N}$ is given by

$$\langle x, y \rangle = \frac{1}{2}(x + y)(x + y + 1) + x$$

where importantly the functions π_1, π_2 such that for all z, $z = \langle \pi_1(z), \pi_2(z) \rangle$ are also computable. We can use primitive recursion to code any finite tuple:

$$\begin{aligned} \lambda &\mapsto \langle 0, 0 \rangle \\ n &\mapsto \langle 0, n+1 \rangle \\ (n_0, \ldots, n_k) &\mapsto \langle k, \langle n_0, \ldots, n_k \rangle \rangle \end{aligned}$$

where λ is the empty sequence.

2.1.3 Universal Register Machines

Definition 2.3. A register machine V is **universal** if for every register machine M there is a number e_M such that for all x, $M(x) \simeq V(e_M, x)$.

Fact. By coding, universal register machines exist.

Fix a universal register machine V. An important consequence of the existence of a universal register machine is that we can put the partial computable functions into an **effective list** $\Phi_0(x), \Phi_1(x), \ldots$, where $\Phi_e(x) = V(e, x)$.

Write $V_s(e, x)$ for the action of V on inputs e and x after s instruction steps. Then write $\Phi_{e,s}(x) = V_s(e, x)$. Assume the following conventions:

- $\Phi_{e,s}(x) \downarrow = y \implies x, y < s$,
- $\Phi_{e,0}(x) \uparrow$
- There is at most one x such that $\Phi_{e,s+1}(x) \downarrow$ but $\Phi_{e,s}(x) \uparrow$ (meaning at each step, Φ_e only converges on at most one new value).

2.1.4 Classic Theorems

Theorem 2.1 (s^m_n Theorem, Basic Version)**.** *There is a computable injective function $s^1_1(e, x)$ such that for all e, x, y, $\Phi_{s^1_1(e,x)}(y) = \Phi_e(x, y)$.*

The s^m_n theorem is usually used to obtain the following type of result:

Example. There is a computable function $h(x)$ such that $\Phi_{h(x)}(y) \downarrow$ if and only if $x = y$.

Proof. Let

$$g(x, y) = \begin{cases} 0 & \text{if } x = y \\ \uparrow & \text{otherwise} \end{cases}.$$

Then g is a partial computable function, so let e be an index for g, meaning $\Phi_e(x, y) = g(x, y)$. By the s^m_n theorem, there is a computable function s^1_1 such that $\Phi_{s^1_1(e,x)}(y) = \Phi_e(x, y) = g(x, y)$. Thus let $h(x) = s^1_1(e, x)$. □

Theorem 2.2 (s^m_n Theorem, Full Version)**.** *For each $m, n \geq 1$, there is a computable injective function s^m_n such that for all n-tuples $\overline{x}$, m-tuples $\overline{y}$, and all e,*

$$\Phi_{s^m_n(e,\overline{x})}(\overline{y}) = \Phi_e(\overline{x}, \overline{y}).$$

Theorem 2.3 (Recursion Theorem)**.** *For every computable function f, there is n such that $\Phi_n = \Phi_{f(n)}$.*

The s^m_n theorem and recursion theorem are often used in combination to obtain the following type of result:

Example. There is n such that $\text{dom}(\Phi_n) = \{n\}$.

Proof. In the previous example, we used the s^m_n theorem to find a function h such that $\Phi_{h(x)}(y)\downarrow$ if and only if $x = y$. Applying the recursion theorem to h, there is n such that $\Phi_n = \Phi_{h(n)}$. Thus $\Phi_n(m)\downarrow$ if and only if $n = m$, meaning $\text{dom}(\Phi_n) = \{n\}$. □

Theorem 2.4 (Halting Problem)**.** *The set $K = \{e : \Phi_e(e)\downarrow\}$ is not computable.*

Proof. Assume towards a contradiction that K is computable, meaning there is an index i such that Φ_i is the characteristic function of K. Define a function f by

$$f(x) = \begin{cases} \Phi_x(x) + 1 & \text{if } \Phi_i(x) = 1 \\ 0 & \text{if } \Phi_i(x) = 0 \end{cases}.$$

Then f is a total computable function, so let e be an index for f. Since Φ_e is total, $\Phi_e(e)\downarrow$, so $\Phi_i(e) = 1$. We then have $f(e) = \Phi_e(e) + 1$, but this is a contradiction since $f(e) = \Phi_e(e)$. Thus the characteristic function for the halting set K cannot be computable, and so the halting problem is not computable. □

2.1.5 Computably Enumerable Sets

Definition 2.4. A set A is **computably enumerable (c.e.)** if there is an index e such that $A = \text{dom}(\Phi_e)$.

Write W_e for $\text{dom}(\Phi_e)$. Thus $W_0, W_1, W_2, \dots$ is an effective list of the c.e. sets.

Example. All computable sets are c.e. The halting set K is a c.e. set which is not computable.

Theorem 2.5. *A set A is c.e. if and only if $A = \text{range}(\Phi_e)$ for some e.*

Theorem 2.6. *A set A is computable if and only if A and its complement are both c.e.*

2.1.6 The Arithmetic Hierarchy

Definition 2.5. Let $n \geq 1$.

- A set $A \subseteq \mathbb{N}$ is Σ^0_n if there is a computable relation $R(x, y_1, \dots, y_n)$ such that

  $$x \in A \iff \exists y_1 \forall y_2 \exists y_3 \cdots \mathsf{Q} y_n R(x, y_1, \dots, y_n)$$

  where the quantifier Q is $\exists$ if n is odd and $\forall$ if n is even.

- A set $A \subseteq \mathbb{N}$ is Π^0_n if there is a computable relation $R(x, y_1, \dots, y_n)$ such that

  $$x \in A \iff \forall y_1 \exists y_2 \forall y_3 \cdots \mathsf{Q} y_n R(x, y_1, \dots, y_n)$$

  where Q is $\forall$ if n is odd and $\exists$ if n is even.

- A set A is Δ^0_n if A is both Σ^0_n and Π^0_n.

Theorem 2.7. *A is Σ^0_1 if and only if A is c.e. Thus A is Π^0_1 if and only if the complement of A is c.e., and A is computable if and only if A is Δ^0_1.*

Example. Here are a few more examples of sets which are higher in the arithmetical hierarchy.

- $\mathrm{Tot} = \{e : \Phi_e \text{ is total}\}$ is Π^0_2, since
$$e \in \mathrm{Tot} \iff \forall x \exists s \Phi_{e,s}(x)\downarrow.$$
- $\mathrm{Fin} = \{e : W_e \text{ is finite}\}$ is Σ^0_2, since
$$e \in \mathrm{Fin} \iff \exists x \forall y \forall s (\Phi_{e,s}(y)\uparrow \vee y \le x).$$
- $\mathrm{Inf} = \{e : W_e \text{ is infinite}\}$ is Π^0_2, since its complement Fin is Σ^0_2.
- $\mathrm{Cof} = \{e : W_e \text{ is cofinite}\}$ is Σ^0_3, since
$$e \in \mathrm{Cof} \iff \exists x \forall y \exists s (y < x \vee \Phi_{e,s}(y)\downarrow).$$
- $\mathrm{CoInf} = \{e : W_e \text{ is coinfinite}\}$ is Π^0_3 since its complement Cof is Σ^0_3.

We showed above that Fin is Σ^0_2. How do we know there is no way to write it more simply, say as a Π^0_1 or Σ^0_1 set? Saying that Fin is Σ^0_2 puts an upper bound on its complexity. The notions of Σ^0_n-completeness and Π^0_n-completeness put lower bounds on the complexities of sets, and we will see that Fin is Σ^0_2-complete, which implies it cannot be written any simpler.

Definition 2.6. Let A and B be sets. A is **1-reducible** to B, written $A \le_1 B$, if there is a computable injective function f such that for all x, $x \in A$ if and only if $f(x) \in B$. Write $A \equiv_1 B$ if $A \le_1 B$ and $B \le_1 A$.

Fact. $\equiv_1$ is an equivalence relation.

Definition 2.7. A set A is **Σ^0_n-complete** if A is Σ^0_n and for all Σ^0_n sets X, $X \le_1 A$. Similarly, A is **Π^0_n-complete** if A is Π^0_n and for all Π^0_n sets X, $X \le_1 A$.

Example. The halting set $K = \{e : \Phi_e(e)\downarrow\}$ is Σ^0_1-complete.

Example. The set $\mathrm{Fin} = \{e : W_e \text{ is finite}\}$ is Σ^0_2-complete.

Proof. We showed above that Fin is Σ^0_2. Let A be Σ^0_2. We must show $A \le_1 \mathrm{Fin}$. Fix a computable relation $R(x,y,z)$ such that $A = \{x : \exists y \forall z R(x,y,z)\}$. Define
$$g(x,u) = \begin{cases} 0 & \text{if } \forall y \le u \exists z \neg R(x,y,z) \\ \uparrow & \text{otherwise} \end{cases}.$$
Fix e such that $\Phi_e(x,u) = g(x,u)$. By the s^m_n theorem, there is a computable injective function s^1_1 such that
$$\Phi_{s^1_1(e,x)}(u) = \Phi_e(x,u) = g(x,u).$$
Let $f(x) = s^1_1(e,x)$. We claim that f is a 1-reduction from A to Fin. Suppose $x \in A$. Then the statement $\exists y \forall z R(x,y,z)$ holds, so fix y_0 such that for all z, $R(x,y,z)$ holds. For all $u \ge y_0$, $g(x,u)\uparrow$, so for all $u \ge y_0$, $\Phi_{f(x)}(u)\uparrow$. Hence $W_{f(x)}$ is finite, so $f(x) \in \mathrm{Fin}$. Now suppose $x \notin A$. Then the statement $\exists y \forall z R(x,y,z)$ does not hold, so for all y, there is z such that $\neg R(x,y,z)$. Then $\Phi_{f(x)}(u)\downarrow$ for all u, meaning $W_{f(x)} = \mathbb{N}$, and $f(x) \notin \mathrm{Fin}$. $\square$

Example. The set $\mathrm{Tot} = \{e : \Phi_e \text{ is total}\}$ is Π^0_2-complete. To prove it, apply the above argument to $\overline{A}$ when A is Π^0_2.

2.1.7 Turing Reducibility

2.2 Randomness

Test

3 Descriptive Set Theory (Spring 2022)

4 Generic Sets and Forcing in Computability (Fall 2022)

5 Weihrauch Degrees (Spring 2023)